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SOME CHARACTERIZATION PROBLEMS IN $\mathcal{LA}$-SEMIHYPERGROUPS

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Abstract

We studied some structural properties of an $\mathcal{LA}$-semihypergroup and generalized/extended the theory of an $\mathcal{LA}$-semigroup in terms of their one-sided ideals. We characterized an intra-regular class of an $\mathcal{LA}$-semihypergroup by using one-sided hyperideals and shown that an $\mathcal{LA}$-semihypergroup is intra-regular, if and only if every left and right hyperideal commute with each other.

1. Introduction

A left almost semigroup ($\mathcal{LA}$-semigroup) is a groupoid $\mathcal{S}$, whose elements satisfy the following left invertive law:

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$$(ab)c = (cb)a, \quad \forall\ a, b, c \in \mathcal{S}.$$

This concept was first given by Kazim and Naseeruddin in 1972 [13]. In an $\mathcal{LA}$- semigroup, the medial law [13] $(ab)(cd) = (ac)(bd)$ holds, $\forall\ a, b, c, d \in \mathcal{S}$. An $\mathcal{LA}$- semigroup may or may not contain a left identity. The left identity of an $\mathcal{LA}$- semigroup allow us to introduce the inverses of elements in an $\mathcal{LA}$- semigroup. If an $\mathcal{LA}$- semigroup contains a left identity, then it is unique [17]. In an $\mathcal{LA}$- semigroup $\mathcal{S}$ with left identity, the paramedial law $(ab)(cd) = (dc)(ba)$ holds, $\forall\ a, b, c, d \in \mathcal{S}$. By using medial law with left identity, we get $a(bc) = b(ac)$, $\forall\ a, b, c \in \mathcal{S}$.

An $\mathcal{LA}$- semigroup is a non-associative and non-commutative algebraic structure mid way between a groupoid and a commutative semigroup. This structure is closely related to a commutative semigroup; indeed if an $\mathcal{LA}$- semigroup contains a right identity, then it becomes a commutative semigroup [17]. The connection between a commutative inverse semigroup and an $\mathcal{LA}$- semigroup was established by Yousafzai et al. in [5] as follows: A commutative inverse semigroup $(\mathcal{S}, .)$ becomes an $\mathcal{LA}$- semigroup $(\mathcal{S}, *)$ under $a * b = ba^{-1}r^{-1}$, $\forall\ a, b, r \in \mathcal{S}$. An $\mathcal{LA}$- semigroup $\mathcal{S}$ with a left identity becomes a semigroup under the binary operation "$\circ_e$" defined as follows: $x \circ_e y = (xe)y$, $\forall\ x, y \in \mathcal{S}$ [6].

There are lot of results which have been added to the theory of an $\mathcal{LA}$- semigroup by Mushtaq, Kamran, Holgate, Jezek, Protic, Madad, Yousafzai and many other researchers. An $\mathcal{LA}$- semigroup is a generalization of a semigroup [17] and it has vast applications in semigroups, as well as in other branches of mathematics.

Hyperstructure theory was introduced in 1934, when Marty [16] defined hypergroups, began to analyze their properties and applied them to groups. In the following decades and nowadays, a number of different hyperstructures are widely studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics by many mathematicians. Nowadays, hyperstructures have a lot of applications to several domains of mathematics and computer science and they are studied in many countries of the world. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. A lot of papers and several books have been written on hyperstructure theory, see [2, 21]. Many authors studied different aspects of semihypergroups, for instance, Bonansinga and Corsini [1]; Davvaz [3]; Drbohlav et al. [4]; Fasino and Freni [7]; Gutan [8]; Hasankhani [9]; Hedayati [10]; Hila et al. [12]; Leoreanu [15]; and Onipchuk [20].

Recently, Hila and Dine [11] introduced the notion of $\mathcal{LA}$-semihypergroups. They investigated several properties of hyperideals of $\mathcal{LA}$-semihypergroup and defined the topological space and study the topological structure of $\mathcal{LA}$-semihypergroups by using hyperideal theory. In [22], Yaqoob et al. have characterized intra-regular $\mathcal{LA}$-semihypergroups by using the properties of their left and right hyperideals, and investigated some useful conditions for an $\mathcal{LA}$-semihypergroup to become an intra-regular $\mathcal{LA}$-semihypergroup.

2. Preliminaries, Examples and Some Important Facts

In this section, we recall certain definitions and results needed for our purpose.

A map $\circ : \mathcal{H} \times \mathcal{H} \to \mathcal{P}^*(\mathcal{H})$ is called *hyperoperation* or *join operation* on the set $\mathcal{H}$, where $\mathcal{H}$ is a non-empty set and $\mathcal{P}^*(\mathcal{H}) = \mathcal{P}(\mathcal{H}) \setminus \{\emptyset\}$ denotes the set of all non-empty subsets of $\mathcal{H}$. A hypergroupoid is a set $\mathcal{H}$ together with a (binary) hyperoperation.

If A and B be two non-empty subsets of $\mathcal{H}$, then we denote

$$A \circ B = \bigcup_{a \in A,\, b \in B} a \circ b, \quad a \circ A = \{a\} \circ A, \text{ and } a \circ B = \{a\} \circ B.$$

A hypergroupoid $(\mathcal{H}, \circ)$ is called an $\mathcal{LA}$- semihypergroup [11] if, $\forall\ x, y, z \in \mathcal{H}$:

$$(x \circ y) \circ z = (z \circ y) \circ x.$$

The law is called a left invertive law.

Every $\mathcal{LA}$- semihypergroup satisfies the following law:

$$(x \circ y) \circ (z \circ w) = (x \circ z) \circ (y \circ w),$$

for all $w, x, y, z \in \mathcal{H}$. This law is known as medial law (cf. [11]).

Definition 1. Let $\mathcal{H}$ be an $\mathcal{LA}$- semihypergroup [22], then an element $e \in \mathcal{H}$ is called

(i) left identity (resp., pure left identity) if, $\forall\ a \in \mathcal{H}, a \in e \circ a$ (resp., $a = e \circ a$);

(ii) right identity (resp., pure right identity) if, $\forall\ a \in \mathcal{H}, a \in a \circ e$ (resp., $a = a \circ e$);

(iii) identity (resp., pure identity) if, $\forall\ a \in \mathcal{H}, a \in e \circ a \bigcap a \circ e$ (resp., $a = e \circ a \bigcap a \circ e$).

An $\mathcal{LA}$- semihypergroup $(\mathcal{H}, \circ)$ with pure left identity e satisfy the following laws [22], $\forall\ w, x, y, z \in \mathcal{H}$.

$$(x \circ y) \circ (z \circ w) = (w \circ z) \circ (y \circ x),$$

called a paramedial law, and

$$x \circ (y \circ z) = y \circ (x \circ z).$$

Example 1 ([22]). Let $\mathcal{H} = \{1, 2, 3, 4, 5\}$ with the binary hyperoperation defined below:

$\circ$	1	2	3	4	5
1	1	1	1	1	1
2	1	$\{3, 5\}$	3	$\{1, 4\}$	$\{3, 5\}$
3	1	3	3	$\{1, 4\}$	3
4	1	$\{1, 4\}$	$\{1, 4\}$	4	$\{1, 4\}$
5	1	$\{2, 5\}$	3	$\{1, 4\}$	$\{2, 5\}$

Clearly $\mathcal{H}$ is not a semihypergroup because $(5 \circ 5) \circ 2 = \{2, 3, 5\} \neq \{2, 5\} = 5 \circ (5 \circ 2)$. Thus $\mathcal{H}$ is an $\mathcal{LA}$- semihypergroup because the elements of $\mathcal{H}$ satisfies the left invertive law.

It is a well known fact that if an $\mathcal{LA}$-semigroup contains a right identity, then it becomes an identity, and an $\mathcal{LA}$- semigroup becomes a commutative semigroup with an identity. But on the other hand, the behaviour of an $\mathcal{LA}$- semihypergroup is much different as compared to an $\mathcal{LA}$-semigroup. It is very interesting and important to note that if an $\mathcal{LA}$-semihypergroup $\mathcal{H}$ contains a right identity, then it need not to be a left identity, which can be seen from the following example:

Example 2. Let $\mathcal{H} = \{1, 2, 3\}$ with the binary hyperoperation defined below:

$\circ$	1	2	3
1	$\{1, 3\}$	3	$\{2, 3\}$
2	$\{2, 3\}$	3	3
3	$\{2, 3\}$	$\{2, 3\}$	$\{2, 3\}$

Clearly $\mathcal{H}$ is not a semihypergroup because $(2 \circ 2) \circ 3 = \{2, 3\} \neq 3 = 2 \circ (2 \circ 3)$. Thus $\mathcal{H}$ is an $\mathcal{LA}$- semihypergroup because the elements of $\mathcal{H}$ satisfies the left invertive law. One can easily observe that 1 is a right identity but not a left identity.

Now, we are going to present another unusual behaviour of an $\mathcal{LA}$-semihypergroup, which is strictly restricted in the structure of an $\mathcal{LA}$-semigroup.

Example 3. Let $\mathcal{H} = \{1, 2, 3\}$ with the binary hyperoperation defined below:

$\circ$	1	2	3
1	$\{1, 3\}$	2	$\{2, 3\}$
2	$\{2, 3\}$	$\{2, 3\}$	$\{2, 3\}$
3	$\{2, 3\}$	$\{2, 3\}$	$\{2, 3\}$

Clearly $\mathcal{H}$ is an $\mathcal{LA}$-semihypergroup because the elements of $\mathcal{H}$ satisfies the left invertive law. It is easy to see that 1 is a right identity and as well as a left identity of $\mathcal{H}$, that is 1 is an identity of $\mathcal{H}$, but $\mathcal{H}$ is neither commutative $[1 \circ 2 = 2 \neq \{2, 3\} = 2 \circ 1]$ nor associative $[(1 \circ 1) \circ 2 = \{2, 3\} \neq 2 = 1 \circ (1 \circ 2)]$.

However, if an $\mathcal{LA}$- semihypergroup contains a pure right identity, then it becomes a commutative semihypergroup with a pure identity [22].

Thus, we have concluded that the structural properties of an $\mathcal{LA}$-semihypergroup are much different than that of an $\mathcal{LA}$-semigroup due to the following major remarks:

Remark 1. The right identity of an $\mathcal{LA}$-semihypergroup need not to be a left identity in general.

Remark 2. An $\mathcal{LA}$-semihypergroup may have a left identity or a right identity or an identity.

Remark 3. An $\mathcal{LA}$-semihypergroup with a right identity need not to be associative.

It is easy to see that if $\mathcal{S}$ is an intra-regular $\mathcal{LA}$-semihypergroup, then $\mathcal{S} = \mathcal{S} \circ \mathcal{S}$ holds [22].

Definition 2. A non-empty subset $\mathcal{A}$ of an $\mathcal{LA}$-semihypergroup $\mathcal{S}$ called left (right) hyperideal of $\mathcal{S}$, if and only if $\mathcal{S} \circ \mathcal{A} \subseteq \mathcal{A}$ $(\mathcal{A} \circ \mathcal{S} \subseteq \mathcal{A})$ and is called two-sided hyperideal or hyperideal of $\mathcal{S}$, if and only if it is both left and right hyperideal of $\mathcal{S}$.

Definition 3. A non-empty subset $\mathcal{A}$ of an $\mathcal{LA}$-semihypergroup $\mathcal{S}$ called semiprime, if and only if $a^2 = a \circ a \in \mathcal{A} \Rightarrow a \in \mathcal{A}$.

3. Main Results

By $\mathcal{R}$ and $\mathcal{L}$, we will mean right and left (hyper-) ideals of an $\mathcal{LA}$-semigroup ($\mathcal{LA}$-semihypergroup) $\mathcal{S}$, respectively, such that $\mathcal{R}$ will be semiprime.

Lemma 1. *An $\mathcal{LA}$-semihypergroup $\mathcal{S}$ with pure left identity is intra-regular, if and only if $\mathcal{R} \cap \mathcal{L} = \mathcal{R} \circ \mathcal{L}$.*

Proof. $(\Rightarrow)$: Suppose that an $\mathcal{LA}$-semihypergroup $\mathcal{S}$ with pure left identity is an intra-regular. Let $\mathcal{R}$ and $\mathcal{L}$ be any right and left hyperideals of $\mathcal{S}$, respectively, and let $a \in \mathcal{S}$, then there exist $x, y \in \mathcal{S}$ such that $a = (x \circ a^2) \circ y$. Now let $a^2 \in \mathcal{R}$, then

$$a = (x \circ a^2) \circ y = ((e \circ x) \circ (a \circ a)) \circ y = ((a \circ a) \circ (x \circ e)) \circ y$$

$$= (y \circ (x \circ e)) \circ (a \circ a) = (a \circ a) \circ ((x \circ e) \circ y) \in \mathcal{R} \circ \mathcal{S} \subseteq \mathcal{R}.$$

Thus $\mathcal{R}$ is semiprime. It is easy to see that $\mathcal{R} \circ \mathcal{L} \subseteq \mathcal{R} \cap \mathcal{L}$. Since $a^2 \in a^2 \circ \mathcal{S}$, as

$$a^2 \in \mathcal{S} \circ a^2 = (\mathcal{S} \circ \mathcal{S}) \circ (a \circ a) = (a \circ a) \circ (\mathcal{S} \circ \mathcal{S}) = a^2 \circ \mathcal{S},$$

and also $a^2 \circ \mathcal{S}$ is a right hyperideal of $\mathcal{S}$, therefore by given assumption $a \in a^2 \circ \mathcal{S}$. Let $a \in \mathcal{R} \cap \mathcal{L}$, then

$$a \in a^2 \circ \mathcal{S} = (a \circ a) \circ (\mathcal{S} \circ \mathcal{S}) = (a \circ \mathcal{S}) \circ (a \circ \mathcal{S}) = (\mathcal{S} \circ a) \circ (\mathcal{S} \circ a)$$

$$= ((\mathcal{S} \circ \mathcal{S}) \circ a) \circ (\mathcal{S} \circ a) = ((a \circ \mathcal{S}) \circ \mathcal{S}) \circ (\mathcal{S} \circ a)$$

$$\subseteq ((\mathcal{R} \circ \mathcal{S}) \circ \mathcal{S}) \circ (\mathcal{S} \circ \mathcal{L}) \subseteq \mathcal{R} \circ \mathcal{L},$$

which shows that $\mathcal{R} \cap \mathcal{L} = \mathcal{R} \circ \mathcal{L}$, where $\mathcal{R}$ and $\mathcal{L}$ are any right and left hyperideals of $\mathcal{S}$, respectively, such that $\mathcal{R}$ is semiprime.

$(\Leftarrow)$: Let every right hyperideal $\mathcal{R}$ be semiprime and $\mathcal{L}$ be any left hyperideal of an $\mathcal{LA}$-semihypergroup $\mathcal{S}$ with pure left identity such that $\mathcal{R} \cap \mathcal{L} = \mathcal{R} \circ \mathcal{L}$. Since $a^2 \circ \mathcal{S}$ and $\mathcal{S} \circ a$ are any right and left hyperideals of $\mathcal{S}$, respectively, then it is easy to see that $a \in a^2 \circ \mathcal{S}$ and $a \in \mathcal{S} \circ a$, therefore,

$$a \in a^2 \circ \mathcal{S} \cap \mathcal{S} \circ a = (a^2 \circ \mathcal{S}) \circ (\mathcal{S} \circ a) = ((a \circ a) \circ (\mathcal{S} \circ \mathcal{S})) \circ (\mathcal{S} \circ a)$$

$$= ((\mathcal{S} \circ \mathcal{S}) \circ (a \circ a))(\mathcal{S} \circ a) \subseteq (\mathcal{S} \circ a^2) \circ \mathcal{S},$$

that is, $a = (x \circ a^2) \circ y$ for some $x, y \in \mathcal{S}$, therefore $\mathcal{S}$ is an intra-regular. □

Corollary 1. *An $\mathcal{LA}$-semigroup $\mathcal{S}$ with left identity is intra-regular, if and only if $\mathcal{R} \cap \mathcal{L} = \mathcal{RL}$.*

Lemma 2 ([22]). *A non-empty subset $\mathcal{A}$ of an intra-regular $\mathcal{LA}$-semihypergroup $\mathcal{S}$ with pure left identity is a left hyperideal of $\mathcal{S}$, if and only if it is a right hyperideal of $\mathcal{S}$.*

Lemma 3. *An $\mathcal{LA}$-semihypergroup $\mathcal{S}$ with pure left identity is intra-regular, if and only if $\mathcal{L} \cap \mathcal{R} = \mathcal{L} \circ \mathcal{R}$.*

Proof. $(\Rightarrow)$ can be followed by using Lemmas 1 and 2. $(\Leftarrow)$ is straightforward. □

Corollary 2. *An $\mathcal{LA}$-semigroup $\mathcal{S}$ with left identity is intra-regular, if and only if* $\mathcal{L} \cap \mathcal{R} = \mathcal{L} \circ \mathcal{R}$.

From above discussion, we get the following immediate results:

Theorem 1. *An $\mathcal{LA}$-semihypergroup $\mathcal{S}$ with pure left identity is intra-regular, if and only if* $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$.

Corollary 3. *An $\mathcal{LA}$-semigroup $\mathcal{S}$ with left identity is intra-regular, if and only if* $\mathcal{RL} = \mathcal{LR}$.

Definition 4. A non-empty subset $\mathcal{A}$ of an $\mathcal{LA}$-semihypergroup $\mathcal{S}$ called idempotent, if and only if $\mathcal{A} = \mathcal{A}^2 = \mathcal{A} \circ \mathcal{A}$.

Corollary 4. *An $\mathcal{LA}$-semihypergroup $\mathcal{S}$ with pure left identity is intra-regular, if and only if every left hyperideal of $\mathcal{S}$ is idempotent.*

Corollary 5. *An $\mathcal{LA}$-semihypergroup $\mathcal{S}$ with pure left identity is intra-regular, if and only if every semiprime right hyperideal of $\mathcal{S}$ is idempotent.*

Corollary 6. *An $\mathcal{LA}$-semigroup $\mathcal{S}$ with left identity is intra-regular, if and only if every left ideal of $\mathcal{S}$ is idempotent.*

Corollary 7. *An $\mathcal{LA}$-semigroup $\mathcal{S}$ with left identity is intra-regular, if and only if every semiprime right ideal of $\mathcal{S}$ is idempotent.*

Theorem 2. *An $\mathcal{LA}$-semihypergroup $\mathcal{S}$ with pure left identity is intra-regular, if and only if* $\mathcal{L} = \mathcal{L}^3$.

Proof. $(\Rightarrow)$: Let $\mathcal{S}$ be an intra-regular $\mathcal{LA}$-semihypergroup with pure left identity and $\mathcal{L}$ be any left hyperideal of $\mathcal{S}$. Then by using Corollary 4, we have

$$\mathcal{L}^3 = \mathcal{L}^2 \circ \mathcal{L} = \mathcal{L} \circ \mathcal{L} \subseteq \mathcal{S} \circ \mathcal{L} \subseteq \mathcal{L}.$$

Now let $a \in \mathcal{L}$, then by using Lemmas 1 and 2, $a^2 \circ \mathcal{S}$ is a left hyperideal of $\mathcal{S}$ such that $a \in a^2 \circ \mathcal{S}$, therefore,

$$a \in a^2 \circ \mathcal{S} = (a \circ a) \circ \mathcal{S} = (\mathcal{S} \circ a) \circ a \subseteq (\mathcal{S} \circ (a^2 \circ \mathcal{S})) \circ a$$

$$= (a^2 \circ (\mathcal{S} \circ \mathcal{S})) \circ a = ((a \circ a) \circ \mathcal{S}) \circ a = ((\mathcal{S} \circ a) \circ a) \circ a$$

$$\subseteq ((\mathcal{S} \circ \mathcal{L}) \circ \mathcal{L}) \circ \mathcal{L} \subseteq (\mathcal{L} \circ \mathcal{L}) \circ \mathcal{L} \subseteq \mathcal{L}^3,$$

which is what we set out to prove.

$(\Leftarrow)$: Let $\mathcal{L}$ be any left hyperideal of an $\mathcal{LA}$-semihypergroup $\mathcal{S}$ with pure left identity such that $\mathcal{L} = \mathcal{L}^3$. Since $\mathcal{S} \circ a$ is left hyperideals of $\mathcal{S}$ and $a \in \mathcal{S} \circ a$, therefore,

$$a \in \mathcal{S} \circ a = ((\mathcal{S} \circ a) \circ (\mathcal{S} \circ a)) \circ (\mathcal{S} \circ a) = ((\mathcal{S} \circ \mathcal{S}) \circ (a \circ a)) \circ (\mathcal{S} \circ a) \subseteq (\mathcal{S} \circ a^2) \circ \mathcal{S},$$

that is, $a = (xa^2)y$ for some $x, y \in \mathcal{S}$, therefore $\mathcal{S}$ is an intra-regular. □

Theorem 3. *For an $\mathcal{LA}$-semihypergroup $\mathcal{S}$ with pure left identity, the following conditions are equivalent*:

(i) $\mathcal{S}$ *is an intra-regular*.

(ii) $\mathcal{L} = \mathcal{L}^{i+1}$, *where* $i = 1, \dots, n$.

Proof. It can be easily followed by generalizing the proof of Theorem 2. □

From the left-right dual of Theorem 3, we have the following theorem:

Theorem 4. *For an $\mathcal{LA}$-semihypergroup $\mathcal{S}$ with pure left identity, the following conditions are equivalent*:

(i) $\mathcal{S}$ *is an intra-regular*.

(ii) $\mathcal{R} = \mathcal{R}^{i+1}$, *where* $i = 1, \dots, n$.

Definition 5. An $\mathcal{LA}$-semihypergroup is called left (right) simple, if and only if it has no proper left (right) hyperideal and is called simple, if and only if it has no proper two-sided hyperideal.

Note that if an $\mathcal{LA}$-semihypergroup $\mathcal{S}$ contains a left identity, then $\mathcal{S} = \mathcal{S}^2$.

Theorem 5. *The following conditions are equivalent for an* $\mathcal{LA}$-*semihypergroup* $\mathcal{S}$ *with pure left identity*:

(i) $a \circ \mathcal{S} = \mathcal{S}$, *for some* $a \in \mathcal{S}$.

(ii) $\mathcal{S} \circ a = \mathcal{S}$, *for some* $a \in \mathcal{S}$.

(iii) $\mathcal{S}$ *is simple*.

(iv) *Any two-sided hyperideal* $\mathcal{A}$ *of* $\mathcal{S}$ *acts as an identity of* $\mathcal{S}$.

(v) $\mathcal{S}$ *is an intra-regular*.

Proof. (i) $\Rightarrow$ (ii) : Let $\mathcal{S}$ be an $\mathcal{LA}$-semihypergroup with pure left identity and assume that $a \circ \mathcal{S} = \mathcal{S}$ holds for some $a \in \mathcal{S}$, then

$$\mathcal{S} = \mathcal{S} \circ \mathcal{S} = (a \circ \mathcal{S}) \circ \mathcal{S} = (\mathcal{S} \circ \mathcal{S}) \circ a = \mathcal{S} \circ a.$$

(ii) $\Rightarrow$ (iii) : Let $\mathcal{S}$ be an $\mathcal{LA}$-semihypergroup with pure left identity such that $a \circ \mathcal{S} = \mathcal{S}$ holds for some $a \in \mathcal{S}$. Suppose that $\mathcal{S}$ is not left simple and let $\mathcal{L}$ be a proper left hyperideal of $\mathcal{S}$, then

$$\begin{aligned}\mathcal{S} \circ \mathcal{L} \subseteq \mathcal{L} \subseteq \mathcal{S} = \mathcal{S} \circ \mathcal{S} = (\mathcal{S} \circ a) \circ \mathcal{S} &= ((\mathcal{S} \circ \mathcal{S}) \circ (e \circ a)) \circ \mathcal{S} \\ &= ((a \circ e) \circ (\mathcal{S} \circ \mathcal{S})) \circ \mathcal{S} = ((a \circ e) \circ \mathcal{S}) \circ (\mathcal{S} \circ \mathcal{S}) \\ &= ((\mathcal{S} \circ e) \circ a) \circ (\mathcal{S} \circ \mathcal{S}) = (\mathcal{S} \circ \mathcal{S}) \circ (a \circ (\mathcal{S} \circ e)) \\ &= a \circ ((\mathcal{S} \circ \mathcal{S}) \circ (\mathcal{S} \circ e)) \subseteq a \circ \mathcal{S},\end{aligned}$$

implies that $s \circ l = a \circ t$, for some $a, s, t \in \mathcal{S}$ and $l \in \mathcal{L}$. Since $s \circ l \in \mathcal{L}$, therefore $a \circ t \in \mathcal{L}$, but $a \circ t \in a\mathcal{S}$. Thus $a \circ \mathcal{S} \subseteq \mathcal{L}$ and therefore, we have

$$\mathcal{S} = a \circ \mathcal{S} \subseteq \mathcal{L},$$

implies that $\mathcal{S} = \mathcal{L}$, which contradicts the given assumption. Thus $\mathcal{S}$ is left simple and similarly we can show that $\mathcal{S}$ is right simple, which shows that $\mathcal{S}$ is simple.

(iii) $\Rightarrow$ (iv) : Let $\mathcal{S}$ be a simple $\mathcal{LA}$-semihypergroup with pure left identity and let $\mathcal{A}$ be any two-sided hyperideal of $\mathcal{S}$, then $\mathcal{A} = \mathcal{S}$. Therefore, we have

$$\mathcal{A} \circ \mathcal{S} = \mathcal{S} \circ \mathcal{S} = \mathcal{S} \circ \mathcal{A}.$$

(iv) $\Rightarrow$ (v) : Let $\mathcal{S}$ be an $\mathcal{LA}$-semihypergroup with pure left identity such that $\mathcal{A} \circ \mathcal{S} = \mathcal{S} = \mathcal{S} \circ \mathcal{A}$ holds for any two-sided hyperideal $\mathcal{A}$ of $\mathcal{S}$. Since $a^2 \circ \mathcal{S}$ is a right hyperideal of $\mathcal{S}$ and we know that every right hyperideal of $\mathcal{S}$ with pure left identity is two-sided hyperideal of $\mathcal{S}$. Thus, $a^2 \circ \mathcal{S}$ is two-sided hyperideal of $\mathcal{S}$ such that $(a^2 \circ \mathcal{S}) \circ \mathcal{S} = \mathcal{S} = \mathcal{S} \circ (a^2 \circ \mathcal{S})$. Let $a \in \mathcal{S}$, then

$$a \in \mathcal{S} = (a^2 \circ \mathcal{S}) \circ \mathcal{S} = ((a \circ a) \circ (\mathcal{S} \circ \mathcal{S})) \circ \mathcal{S}$$

$$= ((\mathcal{S} \circ \mathcal{S}) \circ (a \circ a)) \circ \mathcal{S} = (\mathcal{S} \circ a^2) \circ \mathcal{S},$$

that is, $a = (x \circ a^2) \circ y$ for some $x, y \in \mathcal{S}$, therefore $\mathcal{S}$ is fully regular.

(v) $\Rightarrow$ (i) : Let $\mathcal{S}$ be an intra-regular $\mathcal{LA}$-semihypergroup with pure left identity and let $a \in \mathcal{S}$, then there exist $x, y \in \mathcal{S}$ such that $a = (x \circ a^2) \circ y$. Thus,

$$a = (x \circ a^2) \circ y = ((e \circ x) \circ (a \circ a)) \circ y = ((a \circ a) \circ (e \circ x)) \circ y$$

$$= (y \circ (e \circ x)) \circ (a \circ a) = a \circ ((y \circ (e \circ x)) \circ a) \in a \circ \mathcal{S},$$

which shows that $\mathcal{S} \subseteq \mathcal{S} \circ a$ and $\mathcal{S} \circ a \subseteq \mathcal{S}$ is obvious. Thus $\mathcal{S} \circ a = \mathcal{S}$ holds for some $a \in \mathcal{S}$. □

Corollary 8. *The following conditions are equivalent for an* $\mathcal{LA}$*-semihypergroup* $\mathcal{S}$ *with pure left identity*:

(i) $a \circ \mathcal{S} = \mathcal{S}$, *for some* $a \in \mathcal{S}$.

(ii) $\mathcal{S} \circ a = \mathcal{S}$, *for some* $a \in \mathcal{S}$.

(iii) $\mathcal{S}$ *is right simple*.

(iv) *Any two-sided hyperideal* $\mathcal{A}$ *of* $\mathcal{S}$ *acts as an identity of* $\mathcal{S}$.

(v) $\mathcal{S}$ *is an intra-regular*.

Theorem 6. *The following conditions are equivalent for an* $\mathcal{LA}$*-semihypergroup* $\mathcal{S}$ *with pure left identity*:

(i) $\mathcal{S}$ *is an intra-regular*.

(ii) $\mathcal{S} \circ a = \mathcal{S} = a \circ \mathcal{S}$, *for some* $a \in \mathcal{S}$.

Proof. It can be easily followed by using Theorem 5. □

4. Conclusion

The main aim of this paper is to provide a platform for characterizing $\mathcal{LA}$-semihypergroups and ordered $\mathcal{LA}$-semihypergroups in terms of fuzzy hyperideals using pure left (right) identity.

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